MATH 579 Exam 4 Solutions

1. For all $n \in \mathbb{N}$, prove that $\sum_k \binom{n}{k}\binom{2n}{n-k} = \binom{3n}{n}$.

   First solution: Apply Thm. 4.7 in the text, with $m = 2n$, $k = n$, $i = k$.

   Second solution: In a class with n women and $2n$ men, we choose n students. There are $\binom{3n}{n}$ ways to do this, or we could instead choose k women and $n-k$ men, for every possible $k \in [0, n]$. The sum counts exactly this, because for k outside this range the summand is zero.
   This is a direct application of Thm 4.7. It is also very similar to Problem 33, and to Problem 3 from the exam two years ago.

2. A northeastern lattice path is a path consisting of $(1, 0)$ and $(0, 1)$ steps. How many such paths are there from $(0, 0)$ to $(10, 10)$ that do not pass through $(1, 1)$?

   We count northeast lattice paths in a rectangle of size $n \times k$. There must be k steps north, and n steps east. Hence the northeast lattice paths are bijective with rearrangements of the word $\underbrace{NN\cdots N}_{k}\underbrace{EE\cdots E}_{n}$, which is counted by the multinomial coefficient $\binom{n+k}{n,k} = \binom{n+k}{k}$.

   First solution: There are $\binom{10+10}{10}$ NE lattice paths altogether. We count how many pass through $(1, 1)$ and subtract. This is a product of the number of paths from $(0, 0)$ to $(1, 1)$ and the number of paths from $(1, 1)$ to $(10, 10)$. Hence the answer is $\binom{20}{10} - \binom{2}{1}\binom{18}{9} = 184756 - 97240 = 87516$.

   Second solution: Desired paths must start EE or NN. The former then has a northeast path from $(2, 0)$ to $(10, 10)$, of which there are $\binom{8+10}{10} = 43758$. The latter has a northeast path from $(0, 2)$ to $(10, 10)$, of which there are $\binom{10+8}{8} = 43758$. Altogether there are $43758 + 43758 = 87516$.
   This is very similar to Problems 19, 23, 24, 31, 32, 50.

3. Which monomial term(s) of $(x + y + z)^{16}$ has the largest coefficient? What is that coefficient?

   If $x^a y^b z^c$ is in the expansion, it must have $a+b+c = 16$. Its coefficient is $\frac{16!}{a!b!c!}$. To maximize the coefficient we must minimize the denominator. Note that if $a > b+1$, then $(a-1)+(b+1)+c = a+b+c = 6$, and $\frac{(a-1)!(b+1)!c!}{a!b!c!} = \frac{b+1}{a} < 1$, so $(a-1)!(b+1)!c! < a!b!c!$ and hence $a!b!c!$ could not have been minimal. Hence, by symmetry, a, b, c must all be either equal or differ by 1. Hence two of them are 5 and one is 6. There are therefore three terms, each with coefficient $\binom{16}{5,5,6}$: $2018016x^5y^5z^6, 2018016x^5y^6z^5, 2018016x^6y^5z^5$.
   This is very similar to Problems 10,11,12,44,45.

4. For all $n \in \mathbb{N}$, calculate $\sum\limits_{k \text{ odd}} \binom{n}{k}3^k$.

   By Newton's binomial theorem, we have $(1+3)^n = \sum\limits_{k}\binom{n}{k}3^k$. Also, $(1-3)^n =$

$\sum_k \binom{n}{k}(-3)^k = \sum_k \binom{n}{k}3^k(-1)^k$. Hence $4^n-(-2)^n = \sum_k \binom{n}{k}3^k(1-(-1)^k) = \sum_{k \text{ odd}} \binom{n}{k}3^k 2$. Hence $\sum_{k \text{ odd}} \binom{n}{k}3^k = \frac{4^n-(-2)^n}{2} = 2^{n-1}(2^n-(-1)^n)$.
This is a direct application of theorems proved in class. It is also very similar to Problems 39,40.

5. For all $k \in \mathbb{Z}$, prove that $\binom{n-1}{k-1}\binom{n}{k+1}\binom{n+1}{k} = \binom{n-1}{k}\binom{n+1}{k+1}\binom{n}{k-1}$.

   First, if $k \leq 0$, then both sides are zero since $\binom{n-1}{k-1} = \binom{n}{k-1} = 0$. Otherwise, $k \geq 1$, and we have $LHS = \frac{(n-1)_{k-1}}{(k-1)!}\frac{(n)_{k+1}}{(k+1)!}\frac{(n+1)_k}{k!}$ and $RHS = \frac{(n-1)_k}{k!}\frac{(n+1)_{k+1}}{(k+1)!}\frac{(n)_{k-1}}{(k-1)!}$. Note that both denominators are $(k-1)!k!(k+1)!$ so it suffices to prove that $A=B$, for $A=(n-1)_{k-1}(n)_{k+1}(n+1)_k$, $B=(n-1)_k(n+1)_{k+1}(n)_{k-1}$. These have gcd $C=(n-1)_{k-1}(n)_{k-1}(n+1)_k$. We have $A = C(n-(k+1)+2)(n-(k+1)+1) = C(n-k+1)(n-k)$, $B = C(n-1-k+1)(n+1-(k+1)+1) = C(n-k)(n-k+1)$. Hence $A=B$, which proves the theorem.
   This is solved directly from the definition of binomial coefficients; there are many problems that explore this idea.

6. For $m,n \in \mathbb{N}$, prove that $\frac{1}{m!} = \lim_{n\to\infty}\binom{m+n}{n}n^{-m}$.

   We have $\lim_{n\to\infty}\binom{m+n}{n}n^{-m} = \lim_{n\to\infty}\frac{(m+n)!}{m!n!}n^{-m} = \frac{1}{m!}\lim_{n\to\infty}\frac{(m+n)!}{n^m n!} = \frac{1}{m!}\lim_{n\to\infty}\frac{(m+n)_m}{n^m} = \frac{1}{m!}\lim_{n\to\infty}\frac{(m+n)(m+n-1)\cdots(n+1)}{n^m} = \frac{1}{m!}\lim_{n\to\infty}\frac{(m+n)}{n}\frac{(m+n-1)}{n}\cdots\frac{n+1}{n} = \frac{1}{m!}\lim_{n\to\infty}(1+\frac{m}{n})(1+\frac{m-1}{n})\cdots(1+\frac{1}{n})$. This looks bad until you realize that there are m terms in the product, and m is fixed as $n\to\infty$. Hence this equals $\frac{1}{m!}\lim_{n\to\infty}(1+\frac{m}{n})\lim_{n\to\infty}(1+\frac{m-1}{n})\cdots\lim_{n\to\infty}(1+\frac{1}{n}) = \frac{1}{m!}1\times 1\times\cdots\times 1 = \frac{1}{m!}$.

   This theorem was discovered and proved by the great mathematician Leonhard Euler, at the age of 22.

   This is solved directly from the definition of binomial coefficients; there are many problems that explore this idea.

Exam results: High score=84, Median score=68, Low score=52 (before any extra credit)